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Explicit zeta functions for bosonic and fermionic fields on a non-commutative toroidal spacetime†

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Abstract

Explicit formulae for the zeta functions $\zeta_\alpha(s)$ corresponding to bosonic ($\alpha = 2$) and to fermionic ($\alpha = 3$) quantum fields living on a non-commutative, partially toroidal spacetime are derived. Formulae for the most general case of the zeta function associated with a quadratic + linear + constant form (in $\mathbb{Z}$) are obtained. They provide the analytical continuation of the zeta functions in relation to the whole complex s plane, in terms of series of Bessel functions (of fast, exponential convergence), thus being extended Chowla–Selberg formulae. As is well known, this is the most convenient expression that can be found for the analytical continuation of a zeta function; in particular, the residua of the poles and their finite parts are explicitly given. An important novelty is the fact that simple poles show up at $s = 0$, as well as in other places (simple or double, depending on the number of compactified, non-compactified and non-commutative dimensions of the spacetime) where they had never appeared before. This poses a challenge to the zeta-function regularization procedure.

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1. Introduction

For its application in practice, the zeta-function regularization method relies on the existence of quite simple formulae that give the analytical continuation of the zeta function, $\zeta(s)$, from the region of the complex plane extending to the right of the abscissa of convergence, $\text{Re}\, s > s_0$, to the rest of the complex plane [1–3]. These are not only the reflection formula of the corresponding zeta function in each case, but also some other, very fundamental expressions, such as the Jacobi theta-function identity, Poisson's and Plana's resummation formulae and the Chowla–Selberg (CS) formula. However, some of these powerful expressions are often

† This paper is dedicated to Aleix E T, on the unique and promising occasion of his 18th birthday.

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restricted to specific zeta functions, and their explicit derivation is usually quite involved. For instance, until very recently, the CS formula was only known to exist for the homogeneous, two-dimensional Epstein zeta function. Ultimate extensions of it to more general zeta functions in any number of dimensions will be given in section 2 of this paper.

A fundamental property shared by all zeta functions is the existence of a reflection formula. For the Riemann zeta function it is $\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1-s/2)\zeta(1-s)$. For a generic zeta function, $Z(s)$, we may write it as $Z(\omega - s) = F(\omega, s)Z(s)$. This allows for its analytic continuation in a very easy way—which is, in simple cases, the whole story of the zeta-function regularization procedure. However, the analytically continued expression thus obtained is just another series, which again has a slow convergence behaviour, of power series type [4] (actually the same that the original series had, on its convergence domain). Some years ago, Chowla and Selberg [5] found a formula, for the Epstein zeta function in the two-dimensional case, that yields *exponentially quick convergence everywhere*, not just in the reflected domain. They were very proud of it. In [6], a first attempt was made to try to extend this expression to inhomogeneous zeta functions (very important for physical applications, see [7]), but remaining always in *two* dimensions, for this was commonly believed to be an insurmountable restriction of the original formula (see, for instance, [8]). More recently, extensions to an *arbitrary* number of dimensions [9, 10], for both homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases were constructed. However, some of the new formulae (remarkably the ones corresponding to the zero-mass case, e.g. the original CS framework!) are *not explicit*, since they involve solving a rather non-trivial recurrence. (Incidentally, this explains why the CS formula had not been extended to higher-dimensional Epstein zeta functions before.)

In section 2 we shall finish this programme, by providing for the first time explicit, CS-like extended formulae for *all* possible cases involving forms of the very general type: quadratic + linear + constant. This will complete the construction initiated in [9, 10].

In section 3 we will move to specific applications of these formulae in non-commutative field theory. In particular, we will obtain the explicit analytic continuation of the zeta functions corresponding to scalar and vector fields defined on a quite general, partially non-commutative toroidal manifold. Their pole structure will be discussed in detail. The existence of simple poles at $s = 0$ comes as a novelty in the zeta-function regularization method in this case, confirming a result obtained in [11]. In other places, up to double poles will be shown to appear. The corresponding residua and finite parts at the poles are immediately obtained from these expressions.

2. Extended Chowla–Selberg formulae, associated with arbitrary forms of quadratic + linear + constant type

Let A be a positive-definite elliptic ΨDO of positive order $m \in \boldsymbol{R}$, acting on the space of smooth sections of E, an n-dimensional vector bundle over M, a closed n-dimensional manifold. The *zeta function* ζ_A is defined as

$$\zeta_A(s) = \mathrm{tr}\, A^{-s} = \sum_j \lambda_j^{-s} \qquad \mathrm{Re}\, s > \frac{n}{m} \equiv s_0 \tag{1}$$

where $s_0 = \dim M/\mathrm{ord}\, A$ is called the *abscissa of convergence* of $\zeta_A(s)$. Under these conditions, it can be proven that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\boldsymbol{C}$ (regular at $s = 0$), provided that the principal symbol of A (that is $a_m(x, \xi)$) admits a *spectral cut*: $L_\theta = \{\lambda \in \boldsymbol{C};\ \mathrm{Arg}\,\lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\mathrm{Spec}\, A \cap L_\theta = \emptyset$ (Agmon–Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut L_θ. The only possible singularities of $\zeta_A(s)$ are *simple poles* at $s_k = (n-k)/m,\ k = 0, 1, 2, \ldots, n-1, n+1, \ldots$. Kontsevich and Vishik have managed to extend this definition to the case when $m \in \boldsymbol{C}$ (no spectral cut exists) [12].

Consider now the following zeta function (Re $s > p/2$):

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in \boldsymbol{Z}^p}{}' \left[\tfrac{1}{2}\,(\vec{n}+\vec{c})^T\, A\,(\vec{n}+\vec{c}) + q\right]^{-s} \equiv \sum_{\vec{n}\in \boldsymbol{Z}^p}{}' \left[Q\,(\vec{n}+\vec{c}) + q\right]^{-s}. \quad (2)$$

The prime on a summation sign means that the point $\vec{n} = \vec{0}$ is to be excluded from the sum. As we shall see, this is irrelevant when q or some component of $\vec{c}$ is non-zero but, on the contrary, it becomes an inescapable condition in the case when $c_1 = \cdots = c_p = q = 0$. Note that, alternatively, we can view the expression inside the square brackets of the zeta function as a sum of a quadratic, a linear and a constant form, namely, $Q\,(\vec{n}+\vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$.

Our aim is to obtain a formula that gives (the analytic continuation of) this multi-dimensional zeta function in terms of an exponentially convergent series, and which is valid in the whole complex plane, exhibiting the singularities (poles) of the meromorphic continuation—with the corresponding residua—explicitly. The only condition on the matrix A is that it correspond to a (non-negative) quadratic form, which we call Q. The vector $\vec{c}$ is arbitrary, while q will be (for the moment) a positive constant. As we shall see, the solution to this problem will depend very much (its explicit form) on whether q and/or $\vec{c}$ are zero or not. According to this, we will have to distinguish different cases, leading to unrelated final formulae, all to be viewed as different non-trivial extensions of the CS formula (they will be named ECS formulae, and will carry additional tags, for the different cases).

Use of the Poisson resummation formula in equation (2) yields [9, 10]

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}}\, \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\;\Gamma(s)}$$
$$\times \sum_{\vec{m}\in \boldsymbol{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c})\, \left(\vec{m}^{\mathrm{T}} A^{-1}\vec{m}\right)^{s/2-p/4}\, K_{p/2-s}\!\left(2\pi\sqrt{2q\,\vec{m}^{\mathrm{T}} A^{-1}\vec{m}}\right) \quad (3)$$

where K_ν is the modified Bessel function of the second kind and the subindex $1/2$ in $\boldsymbol{Z}^p_{1/2}$ means that, in this sum, only half of the vectors $\vec{m} \in \boldsymbol{Z}^p$ enter. That is, if we take an $\vec{m} \in \boldsymbol{Z}^p$ we must then exclude $-\vec{m}$ (as a simple criterion one can, for instance, select those vectors in $\boldsymbol{Z}^p\backslash\{\vec{0}\}$ whose first non-zero component is positive). Equation (3) fulfills *all* the requirements of a CS formula, but it is very different from the original one, constituting a non-trivial extension to the case of a quadratic + linear + constant form, in any number of dimensions, with the constant term being non-zero. We shall denote this formula, equation (3), by the acronym ECS1.

It is well known that the only pole of this inhomogeneous Epstein zeta function appears, explicitly, at $s = p/2$, where it belongs. Its residue is given by

$$\mathrm{Res}_{s=p/2}\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}\,(\det A)^{-1/2}. \quad (4)$$

2.1. *Limit $q \to 0$*

After some work, one can obtain the limit of expression (3) as $q \to 0$ (for simplicity we also set $\vec{c} = \vec{0}$)

$$\zeta_{A,\vec{0},0}(s) = 2^{1+s}a^{-s}\zeta(2s) + \sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}\,\zeta_{\Delta_{p-1},\vec{0},0}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4}\,\Gamma(s)}$$
$$\times \sum_{\vec{n}_2\in \boldsymbol{Z}^{p-1}}{}' \sum_{n_1=1}^{\infty} \cos\left(\frac{\pi n_1}{a}\vec{b}^{\mathrm{T}}\vec{n}_2\right) n_1^{s-1/2}\,\left(\vec{n}_2^{\mathrm{T}}\Delta_{p-1}\vec{n}_2\right)^{1/4-s/2}$$

$$\times K_{s-1/2}\left(\frac{2\pi n_1}{\sqrt{a}}\sqrt{\vec{n}_2^{\mathrm{T}}\Delta_{p-1}\vec{n}_2}\right). \qquad (5)$$

In equations (3) and (5), A is a $p \times p$ symmetric matrix $A = \left(a_{ij}\right)_{i,j=1,2,\ldots,p} = A^{\mathrm{T}}$, A_{p-1} the $(p-1)\times(p-1)$ reduced submatrix $A_{p-1} = \left(a_{ij}\right)_{i,j=2,\ldots,p}$, a the first component, $a = a_{11}$, $\vec{b}$ the $p-1$ vector $\vec{b} = (a_{21}, \ldots, a_{p1})^{\mathrm{T}} = (a_{12}, \ldots, a_{1p})^{\mathrm{T}}$ and Δ_{p-1} is the following $(p-1)\times(p-1)$ matrix: $\Delta_{p-1} = A_{p-1} - \frac{1}{4a}\vec{b}\otimes\vec{b}$. More precisely, what one actually obtains by taking the limit is *the reflected formula*, as one would have after using the Epstein zeta-function reflection $\Gamma(s)Z(s; A) = \pi^{2s-p/2}(\det A)^{-1/2}\Gamma(p/2 - s)Z(p/2 - s; A^{-1})$, $Z(s; A)$ being the Epstein zeta function [13]. Finally. it can be written as (5). (It is a rather non-trivial exercise to perform this calculation.) Note that equation (5) has *all* the properties demanded from a CS formula, but it is actually *not explicit*. It is in fact a recurrence, rather lengthy to solve as it stands. In fact, it can be viewed as the *straightforward extension* of the original CS formula to higher dimensions. It was the main result of previous work on this subject, for the case $q = c_1 = \cdots = c_p = 0$ [9, 10].

Using a different strategy, this recurrence will be now solved explicitly, in a much simpler way. Indeed, let us proceed in a complementary way, namely, by performing the inversion provided by the Poisson resummation formula (or the Jacobi identity), with respect to $p - 1$ of the indices (say, $j = 2, 3, \ldots, p$). This leaves us with three sums, corresponding to positive, zero and negative values of the remaining index (n_1, in this case). The zero value of n_1 (in correspondence with the rest of the n_i not being all zero) classifies the number of different situations (according to the values of the c_i and q being all zero or not) into just two cases. (As is immediate, from the start all c_i can be taken to be between zero and unity: $0 \leqslant c_i < 1$, $i = 1, 2, \ldots, p$.) (i) The first case is, thus, when at least one of the c_i or $q \geqslant 0$ is not zero. Since the case $q \neq 0$ has been solved already, we will mean by this case now that, say, $c_1 \neq 0$. (ii) The second case is when all $q = c_1 = \cdots = c_p = 0$.

2.2. *Case with $q = 0$ but $c_1 \neq 0$*

2.2.1. General (non-diagonal) subcase. By performing inversion provided by the Poisson resummation formula (or the Jacobi identity), with respect to $p - 1$ of the indices (here, $j = 2, 3, \ldots, p$), we readily obtain

$$\begin{aligned}\zeta_{A_p,\vec{c},0}(s) = \frac{2^s}{\Gamma(s)}\left(\det A_{p-1}\right)^{-1/2} &\Big\{\pi^{(p-1)/2}\left(a_{11} - \vec{a}_{p-1}^{\mathrm{T}}A_{p-1}^{-1}\vec{a}_{p-1}\right)^{(p-1)/2-s} \\ &\times\Gamma\left(s-(p-1)/2\right)\left[\zeta_H(2s-p+1, c_1) + \zeta_H(2s-p+1, 1-c_1)\right] \\ &+4\pi^s\left(a_{11} - \vec{a}_{p-1}^{T}A_{p-1}^{-1}\vec{a}_{p-1}\right)^{(p-1)/4-s/2} \\ &\times\sum_{n_1\in Z}\sum_{\vec{m}\in Z_{1/2}^{p-1}}{}'\cos\left[2\pi\vec{m}^{\mathrm{T}}\left(\vec{c}_{p-1} + A_{p-1}^{-1}\vec{a}_{p-1}(n_1+c_1)\right)\right] \\ &\times|n_1+c_1|^{(p-1)/2-s}\left(\vec{m}^{T}A_{p-1}^{-1}\vec{m}\right)^{s/2-(p-1)/4} \\ &\times K_{(p-1)/2-s}\left(2\pi|n_1+c_1|\sqrt{\left(a_{11} - \vec{a}_{p-1}^{\mathrm{T}}A_{p-1}^{-1}\vec{a}_{p-1}\right)\vec{m}^{T}A_{p-1}^{-1}\vec{m}}\right)\Big\} \\ &-\left(\tfrac{1}{2}\vec{c}^{T}A\vec{c}\right)^{-s}. \end{aligned} \qquad (6)$$

Here, and in what follows, A_{p-1} is (as before) the submatrix of A_p composed of the last $p-1$ rows and columns. Moreover, a_{11} is the first diagonal component of A_p, while $\vec{a}_{p-1} = (a_{12}, \ldots, a_{1p})^{\mathrm{T}} = (a_{21}, \ldots, a_{p1})^{\mathrm{T}}$ and $\vec{m} = (n_2, \ldots, n_p)^{\mathrm{T}}$. Note that this is an *explicit formula*, that the only pole at $s = p/2$ also appears explicitly and that the second term of the rhs is a series of exponentially fast convergence. It has, therefore (as equation (3)), all the properties required to qualify as a CS formula. We shall name this expression ECS2.

2.2.2. Diagonal subcase. In this very common, particular case the preceding expression reduces to the simpler form:

$$\begin{aligned}\zeta_{A_p,\vec{c},0}(s) = \frac{2^s}{\Gamma(s)} \left(\det A_{p-1}\right)^{-1/2} \Big\{ & \pi^{(p-1)/2} a_1^{(p-1)/2-s} \Gamma\big(s-(p-1)/2\big) \\ & \times \left[\zeta_H(2s-p+1,c_1)+\zeta_H(2s-p+1,1-c_1)\right] + 4\pi^s a_1^{(p-1)/4-s/2} \\ & \times \sum_{n_1\in \boldsymbol{Z}} \sum_{\vec{m}\in \boldsymbol{Z}_{1/2}^{p-1}}{}' \cos\left(2\pi \vec{m}^{\mathrm{T}}\vec{c}_{p-1}\right) |n_1+c_1|^{(p-1)/2-s} \left(\vec{m}^T A_{p-1}^{-1}\vec{m}\right)^{s/2-(p-1)/4} \\ & \times K_{(p-1)/2-s}\left(2\pi |n_1+c_1| \sqrt{a_1 \vec{m}^T A_{p-1}^{-1}\vec{m}}\right) \Big\} - \left(\tfrac{1}{2}\vec{c}^T A\vec{c}\right)^{-s}. \end{aligned} \tag{7}$$

We shall call this formula ECS2d.

2.3. Case with $c_1 = \cdots = c_p = q = 0$

2.3.1. General (non-diagonal) subcase. As remarked in [9, 10], we had not yet been able to obtain a closed formula, but just a (rather non-trivial) recurrence, equation (5), relating the p-dimensional case to the $(p-1)$-dimensional one. After a second look, we have now realized that we can actually still proceed as if we had in fact $c_1 = 1 \neq 0$, both for positive and for negative values of n_1. A sum, though, remains with $n_1 = 0$—and the rest of the n_i not all being zero—which yields, once more, the same zeta function as at the beginning, but corresponding to $p-1$ indices. All in all,

$$\begin{aligned}\zeta_{A_p,\vec{0},0}(s) = \zeta_{A_{p-1},\vec{0},0}(s) + \frac{2^{1+s}}{\Gamma(s)} \left(\det A_{p-1}\right)^{-1/2} \Big\{ & \pi^{(p-1)/2} \left(a_{11} - \vec{a}_{p-1}^{\mathrm{T}} A_{p-1}^{-1}\vec{a}_{p-1}\right)^{(p-1)/2-s} \\ & \times \Gamma\big(s-(p-1)/2\big)\, \zeta_{\mathrm{R}}(2s-p+1) + 4\pi^s \left(a_{11} - \vec{a}_{p-1}^{T} A_{p-1}^{-1}\vec{a}_{p-1}\right)^{(p-1)/4-s/2} \\ & \times \sum_{n=1}^{\infty} \sum_{\vec{m}\in \boldsymbol{Z}_{1/2}^{p-1}}{}' \cos\left(2\pi \vec{m}^{\mathrm{T}} A_{p-1}^{-1}\vec{a}_{p-1} n\right) n^{(p-1)/2-s} \left(\vec{m}^T A_{p-1}^{-1}\vec{m}\right)^{s/2-(p-1)/4} \\ & \times K_{(p-1)/2-s}\left(2\pi n \sqrt{\left(a_{11} - \vec{a}_{p-1}^{\mathrm{T}} A_{p-1}^{-1}\vec{a}_{p-1}\right)\vec{m}^T A_{p-1}^{-1}\vec{m}}\right) \Big\}. \end{aligned} \tag{8}$$

This is also a recurrent expression, an alternative to (5), obtained with the help of a different strategy.

Remarkably enough, it is easy to resolve this recurrence explicitly, and indeed to obtain a *closed formula* for this case (we shall write the dimensions of the submatrices of A as subindices), the result being

$$\begin{aligned}\zeta_{A_p}(s) \equiv \zeta_{A_p,\vec{0},0}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=1}^{p} \left(\det A_{p-j}\right)^{-1/2} \Big\{ & \pi^{(p-j)/2} \left(a_{jj} - \vec{a}_{p-j}^{\mathrm{T}} A_{p-j}^{-1}\vec{a}_{p-j}\right)^{(p-j)/2-s} \\ & \times \Gamma\big(s-(p-j)/2\big)\, \zeta_{\mathrm{R}}(2s-p+j) + 4\pi^s \left(a_{jj} - \vec{a}_{p-j}^{T} A_{p-j}^{-1}\vec{a}_{p-j}\right)^{(p-j)/4-s/2} \\ & \times \sum_{n=1}^{\infty} \sum_{\vec{m}_{p-j}\in \boldsymbol{Z}_{1/2}^{p-j}}{}' \cos\left(2\pi \vec{m}_{p-j}^{\mathrm{T}} A_{p-j}^{-1}\vec{a}_{p-j} n\right) n^{(p-j)/2-s} \end{aligned}$$

$$\times \left(\vec{m}_{p-j}^T A_{p-j}^{-1} \vec{m}_{p-j}\right)^{s/2-(p-j)/4}$$
$$\times K_{(p-j)/2-s}\left(2\pi n \sqrt{\left(a_{jj} - \vec{a}_{p-j}^{\mathrm{T}} A_{p-j}^{-1} \vec{a}_{p-j}\right) \vec{m}_{p-j}^T A_{p-j}^{-1} \vec{m}_{p-j}}\right)\Bigg\}. \qquad (9)$$

With a similar notation as above, here A_{p-j} is the submatrix of A_p composed of the last p–j rows and columns. Moreover, a_{jj} is the jth diagonal component of A_p, while $\vec{a}_{p-j} = (a_{jj+1}, \ldots, a_{jp})^{\mathrm{T}} = (a_{j+1j}, \ldots, a_{pj})^{\mathrm{T}}$ and $\vec{m}_{p-j} = (n_{j+1}, \ldots, n_p)^{\mathrm{T}}$. Again, this is an extension of the CS formula to the case in question. It exhibits all the same good properties. Physically, it corresponds to the homogeneous, massless case. It is to be viewed, in fact, as *the* genuine multi-dimensional extension of the CS formula. We shall call it ECS3.

2.3.2. Diagonal subcase. Let us particularize once more to the diagonal case, with $\vec{c} = \vec{0}$, which is quite important in practice and gives rise to simpler expressions. For the recurrence, we have

$$\zeta_{A_p}(s) = \zeta_{A_{p-1}}(s) + \frac{2^{1+s}}{\Gamma(s)} \left(\det A_{p-1}\right)^{-1/2}$$
$$\times\Bigg[\pi^{(p-1)/2} a_1^{(p-1)/2-s} \Gamma(s-(p-1)/2)\, \zeta_{\mathrm{R}}(2s-p+1) + 4\pi^s a_1^{(p-1)/4-s/2}$$
$$\times \sum_{n=1}^{\infty} \sum_{\vec{m}\in \boldsymbol{Z}^{p-1}}{}' n^{(p-1)/2-s} (\vec{m}^T A_{p-1}^{-1} \vec{m})^{s/2-(p-1)/4}$$
$$\times K_{(p-1)/2-s}\left(2\pi n \sqrt{a_1 \vec{m}^T A_{p-1}^{-1} \vec{m}}\right)\Bigg]. \qquad (10)$$

As above, we can solve this finite recurrence and obtain the following simple and explicit formula for this case:

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} \left(\det A_j\right)^{-1/2} \Bigg[\pi^{j/2} a_{p-j}^{j/2-s} \Gamma(s-j/2)\, \zeta_{\mathrm{R}}(2s-j) + 4\pi^s a_{p-j}^{j/4-s/2}$$
$$\times \sum_{n=1}^{\infty} \sum_{\vec{m}_j\in \boldsymbol{Z}^j}{}' n^{j/2-s} (\vec{m}_j^t A_j^{-1} \vec{m}_j)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\Bigg] \qquad (11)$$

with $A_p = \mathrm{diag}(a_1, \ldots, a_p)$, $A_j = \mathrm{diag}(a_{p-j+1}, \ldots, a_p)$, $\vec{m}_j = (n_{p-j+1}, \ldots, n_p)^{\mathrm{T}}$ and ζ_{R} the Riemann zeta function. Note again the fact that this and equation (9) are *explicit* expressions for the multi-dimensional, generalized CS formula and, in this way, they go beyond any result obtained previously. We name this formula ECS3d.

It is immediate to see that the term for $j = 0$ in the sum yields the last term, $\zeta_{A_1}(s)$, of the recurrence, that is

$$\zeta_{A_1}(s) = \sum_{n_p=-\infty}^{+\infty}{}' \left(\frac{a_p}{2} n_p^2\right)^{-s} = 2^{1+s} a_p^{-s} \zeta_{\mathrm{R}}(2s). \qquad (12)$$

It exhibits a pole, at $s = 1/2$, which is spurious—it is actually *not* a pole of the whole function (since it cancels, in fact, with another one originating from the next term, with further cancellations of this kind going on, each term with the next). Concerning the pole structure of the resulting zeta function, as given by equation (11), it is not difficult to see that *only* the pole at $s = p/2$ is actually there (as it should be). It is in the last term, $j = p - 1$, of the sum, and it has the correct residue, namely

$$\text{Res}\ \zeta_{A_p}(s)\Big|_{s=p/2} = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} \left(\det A_p\right)^{-1/2}. \tag{13}$$

The rest of the apparent poles at $s = (p - j)/2$ are not such: they compensate among themselves, one term of the sum with the next, adding pairwise to zero.

Summing up, this formula, equation (11), provides a convenient analytic continuation of the zeta function to the whole complex plane, with its only simple pole showing up explicitly. Aside from this, the finite part of the first sum in the expression is quite easy to obtain, and the remainder—an awful looking multiple series—is in fact an extremely fast-convergent sum, yielding a small contribution, as happens in the CS formula. In fact, since it corresponds to the case $q = 0$, this expression should be viewed as *the* extension of the original CS formula—for the zeta function associated with an homogeneous quadratic form in two dimensions—to an arbitrary number, p, of dimensions. The rest of the formulae above provide also extensions of the original CS expression.

The general case of a quadratic + linear + constant form has been here thus completed. As we clearly see, the main formulae corresponding to the three different subcases, namely ECS1 equation (3), ECS2 equation (6) and ECS3 equation (9), are in fact quite distinct and one cannot directly go from one to another by adjusting some parameters.

For the sake of completeness, we must mention the following. Notice that all cases considered here correspond to having a non-identically-zero quadratic form Q. For Q identically zero, that is, the linear + constant (or affine) case, the formulae for the analytic continuation are again quite different from the ones above. The corresponding zeta function is called Barnes' zeta function. This case has been throughly studied in [10].

3. Spectral zeta function for both scalar and vector fields on a spacetime with a non-commutative toroidal part

We shall now consider the physical example of a quantum system consisting of scalars and vector fields on a D-dimensional non-commutative manifold, M, of the form $\boldsymbol{R}^{1,d} \otimes \boldsymbol{T}_\theta^p$ (thus $D = d + p + 1$). $\boldsymbol{T}_\theta^p$ is a p-dimensional non-commutative torus, its coordinates satisfying the usual relation: $[x_j, x_k] = \mathrm{i}\theta\sigma_{jk}$. Here σ_{jk} is a real non-singular, antisymmetric matrix of ± 1 entries, and θ is the non-commutative parameter.

This physical system has attracted much interest recently, in connection with M-theory and with string theory [14–20], and also because of the fact that those are perfectly consistent theories by themselves, which could lead to brand new physical situations. It has been shown, in particular, that non-commutative gauge theories describe the low-energy excitations of open strings on D-branes in a background Neveu–Schwarz two-form field [14–16].

This interesting system provides us with a quite non-trivial case where the formulae derived above are indeed useful. For one, the zeta functions corresponding to bosonic and fermionic operators in this system are of a different kind, never considered before. Moreover, they can be most conveniently written in terms of the zeta functions in the previous section. What is also nice is the fact that a unified treatment (with just *one* zeta function) can be given for both cases, the nature of the field appearing there as a simple parameter, together with those corresponding to the numbers of compactified, non-compactified and non-commutative dimensions of the spacetime.

3.1. Poles of the zeta function

The spectral zeta function for the corresponding (pseudo-) differential operator can be written in the form [11]

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s-(d+1)/2)}{\Gamma(s)} \sum_{\vec{n}\in \boldsymbol{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s} \left[1 + \Lambda\theta^{2-2\alpha} Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s} \qquad (14)$$

where $V = \mathrm{Vol}\,(\boldsymbol{R}^{d+1})$, the volume of the non-compact part, and $Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$, a diagonal quadratic form, being the compactification radii $R_j = a_j^{-1/2}$. Moreover, the value of the parameter $\alpha = 2$ for scalar fields and $\alpha = 3$ for vectors distinguishes between the two different fields. In the particular case when we set all the compactification radii equal to R, we obtain

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s-(d+1)/2)}{\Gamma(s) R^{d+1-2s}} \sum_{\vec{n}\in \boldsymbol{Z}^p}{}' I(\vec{n})^{(d+1)/2-s} \left[1 + \Lambda\theta^{2-2\alpha} R^{2\alpha} I(\vec{n})^{-\alpha}\right]^{(d+1)/2-s} \qquad (15)$$

the quadratic form now being $I(\vec{n}) = \sum_{j=1}^p n_j^2$.

After some calculations, this zeta function can be written in terms of the Epstein zeta function of the previous section, with the result

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^\infty \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s)} (-\Lambda\theta^{2-2\alpha})^l\, \zeta_{Q,\vec{0},0}(s+\alpha l-(d+1)/2) \qquad (16)$$

which reduces, in the particular case of equal radii, to

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^\infty \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s)} (-\Lambda\theta^{2-2\alpha})^l\, \zeta_E(s+\alpha l-(d+1)/2) \qquad (17)$$

where we use here the notation $\zeta_E(s) \equiv \zeta_{I,\vec{0},0}(s)$; that is, the Epstein zeta function for the standard quadratic form.

The pole structure of the resulting zeta function deserves a careful analysis. It differs, in fact, very much from all cases that were known in the literature until now. This is not difficult to understand, from the fact that the pole of the Epstein zeta function at $s = p/2 - \alpha k + (d+1)/2 = D/2 - \alpha k$, when combined with the poles of the gamma functions, yields a very rich pattern of singularities for $\zeta_\alpha(s)$, on taking into account the different possible values of the parameters involved. The pole structure is straightforwardly found from the explicit expressions for the zeta functions in section 2.

Having already given the formula (16) above—that contains everything needed to perform such a calculation of pole position, residua and finite part—for its importance for the calculation of the determinant and the one-loop effective action from the zeta function, we will here start by specifying what happens at $s = 0$. Remarkably enough, a pole appears in many cases (depending on the values of the different parameters). This will also serve as an illustration of what one has to expect for other values of s. The general case will be left for the following section.

It is convenient to classify the different possible subcases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$, the pole structure given in table 1.

In table 1, l is the summation index in equation (16). The appearance of a pole of the zeta function $\zeta_\alpha(s)$, for both values of α, at $s = 0$ is, let us repeat, an absolute novelty, bound to have important physical consequences for the regularization process. It is necessary to observe that this fact is *not* in contradiction with the well known theorems on the pole structure of an

Table 1. Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means *multiple* of 2α).

$$\text{For } d = 2k \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = 0 \\ \text{if } D = \dot{\overline{2\alpha}} & \Longrightarrow \quad \zeta_\alpha(0) = \text{finite.} \end{cases}$$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} & \left.\begin{cases} \text{finite} & \text{for } l \leqslant k \\ 0 & \text{for } l > k \end{cases}\right\} \Longrightarrow \zeta_\alpha(0) = \text{finite} \\ \text{if } D = 2\alpha l & \left.\begin{cases} \text{pole} & \text{for } l \leqslant k \\ \text{finite} & \text{for } l > k \end{cases}\right\} \Longrightarrow \zeta_\alpha(0) = \text{pole.} \end{cases}$$

(elliptic) differential operator [21]. The situation that appears in the non-commutative case is completely different. (i) To begin with, we no longer have a standard differential operator, but a strictly *pseudodifferential* one, from the beginning. (ii) Moreover, the new spectrum is *not* perturbatively connected (for $\theta \to 0$) with the corresponding one for the commutative case.

3.2. *Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, in the complex s plane*

Substituting the corresponding formula, from the preceding section, for the Epstein zeta functions in equation (16), we obtain the following explicit analytic continuation of $\zeta_\alpha(s)$ ($\alpha = 2, 3$), for bosonic and fermionic fields, to the *whole* complex s plane:

$$\begin{aligned} \zeta_\alpha(s) = {} & \frac{2^{s-d}\, V}{(2\pi)^{(d+1)/2}\Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)} \left(-2^\alpha \Lambda \theta^{2-2\alpha}\right)^l \sum_{j=0}^{p-1} \left(\det A_j\right)^{-1/2} \\ & \times \Bigg[\pi^{j/2} a_{p-j}^{-s-\alpha l+(d+j+1)/2} \Gamma(s+\alpha l-(d+j+1)/2)\zeta_{\mathrm{R}}(2s+2\alpha l-d-j-1) \\ & + 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbf{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l} \\ & \times \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \Bigg]. \end{aligned} \tag{18}$$

As discussed in the previous section in detail, the non-spurious poles of this zeta function are to be found in the terms corresponding to $j = p - 1$. With the knowledge we have gained from the analytical continuation of the Epstein zeta functions in section 2, the final analysis can be here completed at once. Note that the situation here corresponds to the case of section 2.3.2, namely, the diagonal case with $c_1 = \cdots = c_p = q = 0$.

To be remarked again is that what we have in the end, by using our method, is an exponentially fast-convergent series of Bessel functions together with a first, finite part, where a pole (simple or double, as we shall see) may show up, for specific values of the dimensions of the different parts of the manifold, depending also on the nature (scalar versus vectorial) of the fields (the value of α, see table 1 and equation (18)).

To summarize the discussion at the end of section 2, the pole structure of equation (18) is in fact best seen from equation (16) (for $s = 0$ it has been analysed in table 1 already). For a fixed value of the summation index l, the contribution to the only pole of the zeta function $\zeta_E(s + \alpha l - (d + 1)/2)$, at $s = D/2 - \alpha l$, comes from the last term of the j sum only, namely

Table 2. General pole structure of the zeta function $\zeta_\alpha(s)$, according to the different possible values of D and p being odd or even. In italics, the type of behaviour corresponding to lower values of l is quoted, while the behaviour shown in roman characters corresponds to larger values of l.

	D even	D odd
p odd	(1a) *pole*/finite ($l \geqslant l_1$)	(2a) *pole*/pole
p even	(1b) *double pole*/pole ($l \geqslant l_1, l_2$)	(2b) *pole*/double pole ($l \geqslant l_2$)

from $j = p - 1$. It is easy to check that it yields the corresponding residuum (13). This corresponds to the second sum in equation (18). Combined now with the poles of the gamma functions, and taking into account the first series in l, this yields the following expression for the residua of the zeta function $\zeta_\alpha(s)$ at the poles $s = D/2 - \alpha l$, $l = 0, 1, 2, \ldots$

$$\operatorname{Res} \zeta_\alpha(s)\Big|_{s=D/2-\alpha l} = \frac{2^{p/2-d}\pi^{(p-d-1)/2}V}{\Gamma(p/2)} \left(\det A_p\right)^{-1/2} \frac{(-\Lambda\theta^{2-2\alpha})^l}{l!} \frac{\Gamma(p/2+(1-\alpha)l)}{\Gamma(D/2-\alpha l)} \qquad l = 0, 1, 2, \ldots. \tag{19}$$

Actually, depending on D and p being even or odd, completely different situations arise, for different values of l: from the disappearance of the pole, giving rise to a finite contribution, to the appearance of a simple or a double pole. We shall distinguish four different situations and, to simplify the notation, we will denote by U the whole factor in the expression (19) for the residuum, that multiplies the last fraction of two gamma functions (in short, $\operatorname{Res} \zeta_\alpha = U\,\Gamma_1/\Gamma_2$).

(1) For $D - 2\alpha l = -2h,\ h = 0, -1, -2, \ldots$
   (a) for $p/2 + (1-\alpha)l \neq 0, -1, -2, \ldots \Longrightarrow$ finite,
   $\operatorname{Res} \zeta_\alpha = -h!\, U\, \Gamma(p/2 + (1-\alpha)l)$;
   (b) for $p = 2(\alpha-1)l - 2k$, $k = 0, -1, -2, \ldots \Longrightarrow$ pole,
   $\operatorname{Res} \zeta_\alpha = (h!/k!)\, U$.

(2) For $D - 2\alpha l \neq -2h$, $h = 0, -1, -2, \ldots$
   (a) for $p/2 + (1-\alpha)l \neq 0, -1, -2, \ldots \Longrightarrow$ pole,
   $\operatorname{Res} \zeta_\alpha = U\, \Gamma(p/2 + (1-\alpha)l)/\Gamma(D/2 + \alpha l)$;
   (b) for $p = 2(\alpha-1)l - 2k$, $k = 0, -1, -2, \ldots \Longrightarrow$ double pole,
   $\operatorname{Res} \zeta_\alpha = (-1/k!)\, U\, /\, \Gamma(D/2 + \alpha l)$.

Note that we here just quote the *generic* situation that occurs for *l large* enough in each case. For instance, if $p = 2$ a double pole appears for $l = 1, 2, \ldots$. For $p = 4$, a double pole appears for $l = 1, 2, \ldots$ if $\alpha = 3$, but only for $l = 2, 3, \ldots$ if $\alpha = 2$. For $p = 6$, a double pole appears for $l = 2, 3, \ldots$ if $\alpha = 3$, but only for $l = 3, 4, \ldots$ if $\alpha = 2$, and so on. The case with both D and p even (which implies d odd) is the most involved one. For $p = 2$ and $D = 4$, for instance, there is a transition from a pole for $l = 0$ corresponding to the zeta-function factor, to a pole for $l = 1$ and higher, corresponding to the gamma function in the numerator (the compensation of the pole of the zeta-function factor with the one from the gamma function in the denominator prevents the formation of a double pole). In any case, the explicit analytic continuation of $\zeta_\alpha(s)$ given by equation (18) contains *all* the information one needs for calculating the poles and corresponding residua in a straightforward way.

The pole structure can be summarized as in table 2.

An application of these formulae to the calculation of the one-loop partition function corresponding to quantum fields at finite temperature, on a non-commutative flat spacetime, will be given elsewhere [22].

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